Supplemental Lecture Notes

CSE 20

August 25, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 25. It should serve as one example of how to write a proof.

Claim: For every set A, $|A| < |\mathcal{P}(A)|$

Proof. Let A be an arbitrary set. If A is finite with cardinality n, we know from our result in class that $|\mathcal{P}(A)| = 2^n$, so $|A| < |\mathcal{P}(A)|$. Thus, suppose A is infinite. To show that $|A| < |\mathcal{P}(A)|$ we will first show $|A| \leq |\mathcal{P}(A)|$, and then we will show that $|A| \neq |\mathcal{P}(A)|$. To show that $|A| \leq |\mathcal{P}(A)|$, we must define a one-to-one function from A to $\mathcal{P}(A)$. Consider the function $f(a) = \{a\}$. Clearly, this is well-defined on the specified domain and codomain. It is also one-to-one, as the function's output is defined uniquely by the input value. Thus, $|A| \leq |\mathcal{P}(A)|$.

It remains to show that $|A| \neq |\mathcal{P}(A)|$. Suppose towards a contradiction that $|A| = |\mathcal{P}(A)|$. Then there exists a bijection $g : A \to \mathcal{P}(A)$. Consider the set D_g defined by the property $a \in D_g \leftrightarrow a \notin g(a)$. First, note that $D_g \subseteq A$, so $D_g \in \mathcal{P}(A)$. Thus, since g is bijection, it must be onto, so there must exist some $d \in A$ with $g(d) = D_g$. There are two cases to consider. First, assume $d \in D_g$. Then $d \in g(d)$ since $D_g = g(d)$. However, our definition of D_g states that $d \notin D_g$, so we have a contradiction. For the second case, assume $d \notin D_g$. Then since $D_g = g(d)$, $d \notin D_g$. However, according to the definition of D_g, this means $d \in D_g$, which is again a contradiction. Thus, in any case we have a contradiction and our initial assumption must have been false: g cannot be onto, so there is no bijection between A and $\mathcal{P}(A)$, so $|A| \neq |\mathcal{P}(A)|$. Thus, we have shown that $|A| < |\mathcal{P}(A)|$.

□

Claim: The set $(0, 1)$ is uncountable.

Proof. We proceed via diagonalization. Assume towards the contrary that $(0, 1)$ is in fact countable. Then there exists a bijection $f : \mathbb{Z}^+ \to (0, 1)$. Call $r_i \in (0, 1)$ the value of f at some positive integer i: $f(i) = r_i$. We can represent each r_i as an infinite sequence of digits following a decimal point: $r_i = 0.d_{i1}d_{i2}d_{i3}\ldots$. Since f is a bijection, for any $d \in (0, 1)$, there exists some $x \in \mathbb{Z}^+$ with $f(x) = d$. We will find a contradiction by constructing an argument d that is *not* in the image of f. Before constructing d we define the following function $g : \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 9\} \to \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 9\}$

$$g(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Now we can define d. Again, since $d \in (0, 1)$, d is a decimal point followed by an infinite sequence of digits d_i: $d = d_1d_2d_3\ldots$. We define each d_i by $d_i = g(f(i)_i)$. Thus, for all $i \in \mathbb{Z}^+$ we have $d \neq f(i)$ – the ith digit of d is guaranteed to differ from the ith digit of $f(i)$. Thus, f is not onto, so we have found a contradiction and the set $(0, 1)$ must be uncountable.

□