Supplemental Lecture Notes

CSE 20

August 24, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 24. It should serve as one example of how to write a proof.

Claim: The integers are countable.

Proof. To show that the integers are countable we define a function $f : \mathbb{Z} \to \mathbb{Z}^+$ and show that it is a bijection. Define f as follows:

$$f(x) = \begin{cases} 2x + 1 & x \geq 0 \\ 2(-x) & x < 0 \end{cases}$$

We claim that f is a bijective function. First, note that f is a well-defined function. If x is non-negative, $f(x) > 0$ and is clearly an integer. Similarly, if x is negative, $f(x)$ is then positive. Thus, f always maps elements of the domain onto the codomain and is thus well-defined. Now, we show that f is one-to-one. Suppose a, b are arbitrary integers with $f(a) = f(b)$. We consider two cases. First, both $f(a)$ and $f(b)$ are even. Then both a and b are negative, and we have $2(-a) = 2(-b)$ so $a = b$. In the second case, both $f(a)$ and $f(b)$ are odd. Then a and b are both non-negative, and $2a+1 = 2b+1$, so $a = b$. Thus, f is one-to-one. It remains to show that f is onto. Let $b \in \mathbb{Z}^+$ be arbitrary. If b is odd, then there exists an integer k with $b = 2k+1$. Note that $f(k) = 2k + 1 = b$. If b is instead even, there exists a positive integer k with $b = 2k$. Note that $f(-k) = 2k = b$. Thus, in any case we can find an integer input mapping to b, so f is onto. Since f is one-to-one and onto it is a bijection, so the integers are countable. $\square$

Claim: If A is a finite set with $\mid A \mid = n$, then $\mid \mathcal{P}(A) \mid = 2^n$.

Proof. First we introduce and prove a lemma:

Lemma: For any non-empty set B, and any $b \in B$, we have $|\{s \mid s \subseteq B, b \in s\}| = |\{s \mid s \subseteq B, b \notin s\}|$.

Proof. Let B be an arbitrary infinite set, and let $b \in B$ be arbitrary. Define $B_1 = \{s \mid s \subseteq B, b \in s\}$ and $B_2 = \{s \mid s \subseteq B, b \notin s\}$. We will show that $|B_1| = |B_2|$ be defining a bijection $f : B_1 \to B_2$. Consider $f(t) = t - \{b\}$. First, note that f is a well-defined function; it removes b from a set thus mapping a set in B_1 to one in B_2. Now we claim f is one-to-one. Let s_1, s_2 be arbitrary elements of B_1. If $f(s_1) = f(s_2$, then all their elements are the same without including b. Thus, $s_1 = s_2$ and f is one-to-one. Now we show f is onto. Let s_3 be an arbitrary element of B_2. Then $f(s_3 \cup \{b\}) = s_3$ and f is onto. Thus, f is a bijection and we see $|B_1| = |B_2|$. $\square$

We can now proceed with the original proof. We claim that for all finite sets A with $|A| = n$, we have $|\mathcal{P}(A)| = 2^n$. We will show this by induction on n. In the base case, $n = 0$. The only set of cardinality 0 is $\emptyset$. $\mathcal{P}(\emptyset) = \{\emptyset\}$, and the cardinality of this set is 1. Since $2^0 = 1$, our claim holds. Now we consider the inductive case. Let $k \geq 0$ be an arbitrary integer. Assume that for any set A with cardinality k, then $|\mathcal{P}(A)| = 2^k$. Now consider an arbitrary set $X = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$. We compute $|\mathcal{P}(X)|$. Note that we can partition $\mathcal{P}(X)$ into two disjoint sets: $\{x \mid x \subseteq X, x_{k+1} \in x\}$ and $\{x \mid x \subseteq X, x_{k+1} \notin x\}$. We compute the cardinalities of each one. First, note that $\{x \mid x \subseteq X, x_{k+1} \notin x\} = \mathcal{P}(\{x_1, x_2, \ldots, x_k\})$ since we are interested only in the subsets that do not contain x_{k+1}. Thus, $|\{x \mid x \subseteq X, x_{k+1} \notin x\}| = 2^k$, since the set on the right has k elements. Now we consider $|\{x \mid x \subseteq X, x_{k+1} \notin x\}|$. By our lemma, the cardinality of this set is equal to the cardinality of $\{x \mid x \subseteq X, x_{k+1} \notin x\}$. We can thus compute the total cardinality:

$$
\begin{aligned}
|\mathcal{P}(X)| &= |\{x \mid x \subseteq X, x_{k+1} \in x\}| + |\{x \mid x \subseteq X, x_{k+1} \notin x\}| \\
&= 2^k + 2^k \\
&= 2^{k+1}
\end{aligned}
$$

Clearly, the inductive case holds as well, and for any set A with $\mid A \mid= n$, we see $\mid \mathcal{P}(A) \mid= 2^n$.

$\square$