This document contains (semi) formally written versions of some proofs done in lecture. It should serve as one example of how to write a proof.

We define the set $T$ of Full Binary Trees as follows:

- $r \in T$ is a single vertex
- For any trees $t_1, t_2 \in T$, a root $r$ with left child $t_1$ and right child $t_2$ is also in $T$.

We also define the following functions $E$ and $V$ computing the number of edges and vertices in a full binary tree respectively.

\[
\begin{align*}
E(r) &= 0 & \text{where } r \text{ a single vertex} \\
E(t) &= E(t_1) + E(t_2) + 2 & \text{where } t_1, t_2 \text{ are the left and right children of } t \\
V(r) &= 1 & \text{where } r \text{ a single vertex} \\
V(t) &= V(t_1) + E(t_2) + 1 & \text{where } t_1, t_2 \text{ are the left and right children of } t
\end{align*}
\]

Claim: For all trees $t \in T$, $V(t) = E(t) + 1$

Proof. We proceed by induction. Consider first a tree $r \in T$ consisting of a single root node. By definition, $E(r) = 0$ and $V(r) = 1$ so $V(r) = E(r) + 1$ and the property holds. Now, let $t_1$ and $t_2$ be arbitrary trees with $V(t_1) = E(t_1) + 1$ and $V(t_2) = E(t_2) + 1$. Now, consider $t' \in T$, the tree formed by adding a new vertex with left child $t_1$ and right child $t_2$. We compute the
number of vertices in $t'$:

\[
V(t') = V(t_1) + V(t_2) + 1 \\
= E(t_1) + 1 + E(t_2) + 1 + 1 \quad \text{By the inductive hypothesis} \\
= E(t_1) + E(t_2) + 2 + 1 \\
= E(t') + 1 \quad \text{By definition}
\]

Thus, for any tree $t \in T$ we see that $V(t) = E(t) + 1$. \hfill \Box

Consider the following recursively defined set $S$:

- $3 \in S$
- If $x \in S$, then $2x + 1 \in S$

**Claim:** For all $x \in S$, $x \mod 4 = 3$

**Proof.** We proceed by induction. Consider first the base case: $3 \mod 4 = 3$, so our claim holds for the initial element of $S$. Now, let $x \in S$ be an arbitrary element with $x \mod 4 = 3$. Now we compute $2x + 1 \mod 4$. Since $x \mod 4 = 3$, there exists an integer $y$ with $x = 4y + 3$. Thus, $2x + 1 = 2(4y + 3) + 1 = 8y + 7 = 4(2y + 1) + 3$. Thus, since $2y + 1$ is also an integer, $2x + 1 \mod 4 = 3$ and we see that the property is true of any $x \in S$. \hfill \Box