Supplemental Lecture Notes

CSE 20

August 18, 2021

This document contains (semi) formally written versions of some proofs done in lecture. It should serve as one example of how to write a proof.

We define the set T of Full Binary Trees as follows:

- $r \in T$ is a single vertex
- For any trees $t_1, t_2 \in T$, a root r with left child t_1 and right child t_2 is also in T.

We also define the following functions E and V computing the number of edges and vertices in a full binary tree respectively.

$$
\begin{aligned}
E(r) &= 0 && \text{where } r \text{ a single vertex} \\
E(t) &= E(t_1) + E(t_2) + 2 && \text{where } t_1, t_2 \text{ are the left and right children of } t \\
V(r) &= 1 && \text{where } r \text{ a single vertex} \\
V(t) &= V(t_1) + E(t_2) + 1 && \text{where } t_1, t_2 \text{ are the left and right children of } t
\end{aligned}
$$

Claim: For all trees $t \in T$, $V(t) = E(t) + 1$

Proof. We proceed by induction. Consider first a tree $r \in T$ consisting of a single root node. By definition, $E(r) = 0$ and $V(r) = 1$ so $V(r) = E(r) + 1$ and the property holds. Now, let t_1 and t_2 be arbitrary trees with $V(t_1) = E(t_1) + 1$ and $V(t_2) = E(t_2) + 1$. Now, consider $t' \in T$, the tree formed by adding a new vertex with left child t_1 and right child t_2. We compute the number of vertices in t':

$$\begin{aligned} V(t') &= V(t_1) + V(t_2) + 1 \\ &= E(t_1) + 1 + E(t_2) + 1 + 1 && \text{By the inductive hypothesis} \\ &= E(t_1) + E(t_2) + 2 + 1 \\ &= E(t') + 1 && \text{By definition} \end{aligned}$$

Thus, for any tree $tinT$ we see that $V(t) = E(t) + 1$. □

Consider the following recursively defined set S:

- $3 \in S$
- If $x \in S$, then $2x + 1 \in S$

Claim: For all $x \in S$, $x \mod 4 = 3$

Proof. We proceed by induction. Consider first the base case: $3 \mod 4 = 3$, so our claim holds for the initial element of S. Now, let $x \in S$ be an arbitrary element with $x \mod 4 = 3$. Now we compute $2x + 1 \mod 4$. Since $x \mod 4 = 3$, there exists an integer y with $x = 4y + 3$. Thus, $2x + 1 = 2(4y + 3) + 1 = 8y + 7 = 4(2y + 1) + 3$. Thus, since $2y + 1$ is also an integer, $2x + 1 \mod 4 = 3$ and we see that the property is true of any $x \in S$. □