Supplemental Lecture Notes

CSE 20

August 17, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 16. It should serve as one example of how to write a proof.

Claim: In the two-player game Nim (as defined in the slides), the second player can always guarantee a win when both piles start with n objects, where n is a positive integer.

Proof. We proceed via strong induction. Let $P(n)$ be the proposition that, for some positive integer n, the second player in a two-player game of Nim initialized with two piles of n objects can guarantee victory. We will show that $P(n)$ holds for all positive integers n. First, consider the base case where $n = 1$. The first player has only one legal decision: they take one object from one of the piles. Now, the second player can take one object from the other pile, winning the game. Thus, $P(1)$ holds. Now we claim that, for all integers $k > 1$, $P(1) \land P(2) \land \ldots \land P(k-1) \implies P(k)$. Let $k > 1$ be arbitrary, and assume that the inductive hypothesis holds: $P(1) \land P(2) \land \ldots \land P(k-1)$. We now show $P(k)$. Consider a game of Nim where both piles start with k objects. We consider two cases. First, the first player might take all k objects from one pile. Then, the second player empties the other pile and wins. Second, suppose the first player takes j objects from some pile, where $1 \leq j < k$. Then, the second player should also take j objects from the other pile. Now, both piles have $k - j$ objects, where $1 \leq j \leq k - 1$. By our strong inductive hypothesis, player two has a winning strategy in this new game where both piles have $k - j$ objects remaining. Thus, $P(k)$ holds, and we have shown by strong induction that $P(n)$ holds for all positive integers n. $\square$

Claim: For all $n \geq 0$, $f_n \leq 2^n$, where f_n is the nth Fibonacci number.

Proof. Let $P(n)$ be the proposition that $f_n \leq 2^n$, where f_n is the nth Fibonacci number and $n \geq 0$. We show by strong induction that $\forall n \geq 0 P(n)$. We first consider two base cases: $P(0)$ and $P(1)$. By definition, $f_0 = 0$, and $2^0 = 1$, so $P(0)$ holds. Similarly, $f_1 = 1$ and 2^1 so $P(1)$ also holds. We now consider the inductive case. We claim that for all $k > 1$, $P(0) \wedge P(1) \wedge \ldots \wedge P(k-1) \implies P(k)$. Let $k > 1$ be arbitrary and assume that the strong inductive hypothesis holds. Now, consider f_k :

$$\begin{aligned}
f_k &= f_{k-1} + f_{k-2} \\
&\leq 2^{k-1} + f_{k-2} && \text{By the strong inductive hypothesis} \\
&\leq 2^{k-1} + 2^{k-2} && \text{By the strong inductive hypothesis} \\
&= 2^{k-2}(2+1) \\
&\leq 2^{k-2}(2+2) \\
&= 2^{k-2}4 \\
&= 2^{k-2}2^2 \\
&= 2^k
\end{aligned}$$

Thus, $f_k \leq 2^k$ and $P(k)$ holds. We have shown by strong induction that $P(n)$ is true for all non-negative integers n.

□