Supplemental Lecture Notes

CSE 20

August 16, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 16. It should serve as one example of how to write a proof.

Claim: For all non-negative integers n, $(1+\frac{1}{2})^n \geq 1+\frac{n}{2}$.

Proof. Let $P(n)$ be the proposition that for some integer $n \geq 0$, we have $(1+\frac{1}{2})^n \geq 1+\frac{n}{2}$. We claim that $P(n)$ holds for all non-negative integers n and proceed by induction. First, consider the base case $P(0)$. Substituting 0 for n we see $(1+\frac{1}{2})^0 = 1$ and $1+\frac{0}{2} = 1$. Thus, the two quantities are equal and our inequality holds. Now we consider the inductive case. It remains to show that for all $n \geq 0$, $P(n) \implies P(n+1)$. Let $k \geq 0$ be arbitrary. We assume $P(k)$ holds and wish to show $P(k+1)$:

$$\begin{aligned}
\left(1+\frac{1}{2}\right)^{k+1} &= \left(1+\frac{1}{2}\right)^{k}\left(1+\frac{1}{2}\right) \\
&\geq \left(1+\frac{k}{2}\right)\left(1+\frac{1}{2}\right) && \text{by the inductive hypothesis} \\
&= 1+\frac{k}{2}+\frac{1}{2}+\frac{k}{4} \\
&\geq 1+\frac{k}{2}+\frac{1}{2} \\
&= 1+\frac{k+1}{2}
\end{aligned}$$

Thus, we see $\left(1+\frac{1}{2}\right)^{k+1} \geq 1+\frac{k+1}{2}$ so $P(k)$ implies $P(k+1)$. As we have also shown $P(0)$, we can conclude by induction that $P(n)$ holds for all non-negative integers n.

□

Claim: For all non-negative integers n, $\sum_{i=0}^{n}(2i+1) = (n+1)^2$

Proof. Let $P(n)$ be the proposition that for some non-negative integer n, $\sum_{i=0}^{n}(2i+1) = (n+1)^2$. We wish to show that $P(n)$ holds for all n. We proceed by induction. First, consider $P(0)$. By substitution we see that $\sum_{i=0}^{0}(2i+1) = (0+1) = 0 = (0+1)^2$. It remains to show that for all non-negative integers k, $P(k)$ implies $P(k+1)$. Let k be an arbitrary non-negative integer. Assume $P(k)$ holds. We wish to prove $P(k+1)$. The sum becomes

$$\begin{aligned}
\sum_{i=0}^{k+1}(2i+1) &= \sum_{i=0}^{k}(2i+1) + (2(k+1)+1) \\
&= (k+1)^2 + (2(k+1)+1) && \text{by the inductive hypothesis} \\
&= k^2 + 2k + 1 + 2k + 2 + 1 \\
&= k^2 + 4k + 4 \\
&= (k+2)^2
\end{aligned}$$

Thus, we see that $P(k+1)$ holds. As we have shown that $P(0)$ is true, and that for all non-negative integers k $P(k) \implies P(k+1)$, we can conclude by induction that $P(n)$ holds for all non-negative integers n.

□

Claim: For any non-negative integer n, if A is a set of size n, then for any finite set B we have $|A \times B| = |A|\,|B|$.

Proof. Let $P(n)$ be the proposition that for some non-negative integer n, if A is a set of size n, then for any finite set B we have $|A \times B| = |A|\,|B|$. We will show by induction that $\forall n P(n)$. First, consider $P(0)$. If A is of size 0, then A is the emptyset and for any finite set B, $A \times B$ is also empty and its size is 0. Thus, $P(0)$ holds. We move on to the inductive step, claiming that for all non-negative integers k, $P(k) \implies P(k+1)$. Let $k \geq 0$ be arbitrary. Let $C = \{x_1, x_2, \ldots, x_k\}$ be an arbitrary set of size k. Let B be an arbitrary set of finite size. We assume that $P(k)$ holds, so $|C \times B| = |C|\,|B| = k\,|B|$. Now, let x_{k+1} be an arbitrary element such that $A = C \cup \{x_{k+1}\}$ and A has size $k+1$. Now consider the cartesian product $A \times B$. By definition of the cartesian product, $A \times B = (C \times B) \cup \{(x_{k+1}, b) \mid b \in B\}$. Since $x_{k+1} \notin B$, we know $C \times B$ and $\{(x_{k+1}, b) \mid b \in B\}$ are disjoint. Thus, their intersection is empty, so by the equality on slide 23 we have $|A \times B| = |C \times B| + |B| = |C|\,|B| + |B|$. By the inductive hypothesis we can further simplify this to $k\,|B| + |B| = (k+1)\,|B| = |A|\,|B|$. Thus, $P(k) \implies P(k+1)$. Since we have shown that $P(0)$ is true, and that $P(k) \implies P(k+1)$ for all non-negative integers k, we can infer by induction that $P(n)$ is true for all non-negative integers n.

$\square$