Supplemental Lecture Notes

CSE 20

August 12, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 12. It should serve as one example of how to write a proof.

Slide 4

Claim: Three of any 25 unique chosen days must fall in the same month of the year.

Proof. Suppose to the contrary that we can pick 25 unique days of the year while ensuring that no more than two fall in any given month. There are 12 months, so with a maximum of two days per month this means we picked at most 24 days. This is a contradiction, so our original assumption was false and three days must fall in the same month.

□

Slide 6

Claim: There are infinitely many composite numbers.

Proof. For the purposes of contradiction, assume instead that there are only a finite number of composite numbers. Let $\{C_1, C_2, \ldots, C_n\}$ be the finite set of composites. By definition, we know that all C_i are greater than 1. Now, consider the product $C' = C_1 \times C_2 \times \ldots \times C_n$. Since all $C_i > 1$, we know that $C' > C_i$ for all i. We also know that C' is composite, as it is the product of composite numbers. Thus, C' is composite and it is not include in our original finite set of n composite numbers, so we have found a contradiction and the set of composites must be infinite. □

Slide 7

Claim: There exist irrational numbers x, y where x^y is rational.

Proof. By our proof above, we know that $\sqrt{2}$ is irrational. Consider the quantity $\sqrt{2}^{\sqrt{2}}$. There are two cases: either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not. If it's rational, we can set $x = y = \sqrt{2}$ and we've found irrational x and y such that x^y is rational. Now we consider the other case and assume that $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Both are irrational. In this case, we have $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$. 2 is clearly rational. Since we assumed both x and y are irrational, we have again found satisfying values for x and y. Thus, in all cases it is possible to construct irrational values x and y where x^y is rational.

□

Slide 18

Claim: The set $\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k+2)\}$ equals $\{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\}$

Proof. First, we show that $\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k + 2)\} \subseteq \{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\}$. Let $x \in \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k + 2)\}$ be arbitrary. Then there exists an integer k with $x = 2k + 2$. Let $l = k + 1$. Then $x = 2l$. Therefore $x \in \{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\}$ and our subset relation holds. It remains to show that $\{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\} \subseteq \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k + 2)\}$ Let $y \in \{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\}$ be arbitrary. By definition, there exists an integer l with $y = 2l$. Let $k = l - 1$. Then $l = k + 1$ and $y = 2(k + 1) = 2k + 2$. Thus, $y \in \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k + 2)\}$. and $\{n \in \mathbb{Z} \mid \exists l \in \mathbb{Z}(n = 2l)\} \subseteq \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(n = 2k + 2)\}$. As we have shown set containment in both directions, the two sets are equal.

□

Slide 30

Claim: For any sets A, B, C, if $A \cup B \subseteq C$ then $A \subseteq C$ and $B \subseteq C$.

Proof. Let A, B, C be arbitrary sets such that $A \cup B \subseteq C$. Let $x \in A$ be arbitrary. Since $x \in A$, we know that $x \in A \cup B$. Since $A \cup B \subseteq C$, we can also infer that $x \in C$. Thus, $A \subseteq C$. It remains to show that $B \subseteq C$.

Let $y \in B$ be arbitrary. Since $y \in B$, we also know that $y \in A \cup B$. Since $A \cup B \subseteq C$, we in turn infer that $y \in C$. Since $y \in B$ is arbitrary and $y \in C$, we see that $B \subseteq C$ and our original claim holds. $\square$