Supplemental Lecture Notes

CSE 20

August 11, 2021

This document contains (semi) formally written versions of some proofs done in lecture on August 11. It should serve as one example of how to write a proof.

Slide 7

Claim: For all integers a, b, we have $\max(a, b) + \min(a, b) = a + b$

Proof. Let $a, b \in \mathbb{Z}$ be arbitrary. We consider two cases: either $a \leq b$, or $a > b$. In the first case, we assume $a \leq b$. Then by the definitions of maximum and minimum, we know $\max(a, b) = b$ and $\min(a, b) = a$. Thus, $\max(a, b) + \min(a, b) = a + b$. Now, consider the second case. Suppose instead that $a > b$. In this case, $\max(a, b) = a$ and $\min(a, b) = b$. Again, we know that $\max(a, b) + \min(a, b) = a + b$. In both cases, the sums of the maximum and minimum become $a + b$, so our claim holds. □

Slide 9

Claim: If $n \in \mathbb{Z}$ is not a multiple of 3, then $n^2 - 1$ is a multiple of 3.

Proof. Let $n \in \mathbb{Z}$ be arbitrary. We assume n is not a multiple of 3. We must consider two cases: either $n \mod 3 = 1$ or $n \mod 3 = 2$. First, assume $n \mod 3 = 1$. Then, there exists some $k \in \mathbb{Z}$ with $n = 3k + 1$. We now consider the quantity $n^2 - 1$. We can rewrite this as $(3k + 1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 9k^2 + 6k = 3(3k^2 + 2)$. Since the integers are closed under addition and multiplication, $3k^2 + 2$ is an integer and $n^2 - 1$ is therefore divisible by 3. Now we consider the second case and assume that instead there exists some $k \in \mathbb{Z}$ where $n = 3k + 2$. We now consider again the quantity $n^2 - 1$. We can rewrite this as $(3k+2)^2-1 = 9k^2+12k+4-1 = 9k^2+12k+3 = 3(3k^2+4k+1)$. Since the integers are closed under addition and multiplication, $3k^2+4k+1$ is an integer and n^2-1 is therefore divisible by 3. Since n^2-1 is a multiple of 3 in all cases, we can infer that its a multiple of three whenever n is not. □

Slide 16

Claim: The real number $\sqrt{2}$ is not rational.

Proof. Suppose for the purposes of a contradiction that $\sqrt{2}$ is in fact rational. Then by definition there exist integers p, q such that $\sqrt{2} = \frac{p}{q}$, $q \neq 0$, and p and q do not share any factors. By squaring both sides we get $2 = \frac{p^2}{q^2}$, in turn implying that $2q^2 = p^2$. Thus, p^2 is even since it's divisible by two. We can also infer that p is even from our result in class on August 10. By definition, there exists an integer k with $p = 2k$. Thus, $p^2 = (2k)^2 = 4k^2$. Since we also showed that $2q^2 = p^2$, we now have $2q^2 = 4k^2$, implying that $q^2 = 2k^2$. Thus, q^2 is even and q is in turn even. We have showed that both p and q are even, which is impossible as we assumed that they share no common factors. We have arrived at a contradiction, indicating that our original assumption was false and $\sqrt{2}$ is irrational. □

Slide 18

Claim: For integers $k > 1$, $2^k - 1$ is not a perfect square.

Proof. By way of contradiction assume that there exists some k such that $k > 1$ and $2^k - 1$ is a perfect square. Then there exists an integer m with $2^k - 1 = m^2$. Since $k > 1$, we know that 2^k is even. Thus, $2^k - 1$ is odd and m^2 is also odd. Therefore m is odd as well. By definition, this means that there exists some integer t with $m = 2t+1$. Thus, $m^2 = (2t+1)^2 = 4t^2+4t+1$. Now we have $2^k - 1 = 4t^2+4t+1$, so we also have $2^k = 4t^2+4t+2$. Dividing both sides by 2 we see that $2^{k-1} = 2t^2+2t+1$. Since t^2+t is an integer, and $2t^2+2t+1 = 2(t^2+t)+1$, we know that 2^{k-1} is odd. This is only possible if $k-1 = 0$, indicating that $k = 1$. However, we assumed that $k > 1$, so we have arrived at aa contradiction and $2^k - 1$ must not be a perfect square for all integers $k > 1$.

□