CSE 20 Summer 2021
Homework 4
Due date: Sunday, August 29, 2021 at 11:59pm

TOPICS Cardinalities, countable and uncountable infinity, modular arithmetic

READING

- Sections 2.3, 2.5, 4.1, 4.5, 4.6

KEY CONCEPTS diagonalization, bijections, comparing sizes of infinite sets, Modular arithmetic, Equivalence relations.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (16 points) Consider the set $F = \{S \subseteq \mathbb{Z}^{\geq 0} \mid |S| \text{ is finite}\}$, in other words, the set of all finite subsets of $\mathbb{Z}^{\geq 0}$.

   Consider the function $f : F \to \mathbb{Z}^{\geq 0}$ defined by: $f(\emptyset) = 0$ and for any non-empty set $S = \{x_1, x_2, \ldots, x_n\}$, $f(S) = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$.

   (a) Evaluate the function on the following sets:
      i. $\{0, 1, 2, 3, 4\}$
      ii. $\{8, 10, 20\}$
      iii. $\{0\}$

   (b) For each non-negative integer, find the set that maps to it:
      i. 66
      ii. 127
      iii. 25

   (c) Prove that f is one-to-one. You can use the fact that each non-negative integer has a unique representation in base 2.

   (d) Prove that f is onto. (You can use the fact that we proved in class that each positive integer can be written as a sum of distinct powers of 2.)

   (e) (not for credit:) what can be said about $|F|$?

2. (16 points)

   Prove the following statement using induction:

   For all integers $n \geq 2$, if $S_1, S_2, \ldots, S_n$ are all countably infinite sets, then $S_1 \times S_2 \times \cdots \times S_n$ is also countably infinite.

   You can use without proof the fact that if A and B are both countably infinite sets then $A \times B$ is countably infinite.
   You can also use without proof the fact that:

$$|S_1 \times \cdots \times S_{n-1} \times S_n| = |(S_1 \times \cdots \times S_{n-1}) \times S_n|$$

3. (18 points) For each of the following sets, state whether it is **finite**, **countably infinite**, or **uncountable**. With each statement, justify your answer.

- If you want to prove that a set S is countably infinite, you can describe a way to list out all of its elements in a particular way, or you can show that $|S| = |\mathcal{Z}^+|$.
- If you want to prove that a set S is uncountably infinite, you can use a diagonalization proof, or show that it has the same (or larger) cardinality than a well-known uncountable set.

(a) The set of all functions from $\{0, 1\}$ to $\mathbb{Z}^+$.

(b) The set of all infinite sequences of bits.

(c) The set of all multiples of 10.

(d) The set of all Full Binary Trees.

(e) The set of all functions from $\mathbb{Z}^+$ to $\{0, 1\}$.

(f) The set of all binary relations of $\mathbb{Z}^+$

4. (16 points)

(a) Define the relation $\mathbf{R}$ on $\mathcal{P}(\mathbb{Z}^{\geq 0})$ given by the following rule: $A \mathbf{R} B$ iff $|A| = |B|$.
Prove or disprove that $\mathbf{R}$ is an equivalence relation.

(b) Define the relation $\mathbf{J}$ on $\mathcal{P}(\mathbb{Z}^{\geq 0})$ given by the following rule: $A \mathbf{J} B$ iff $A \cap B = \emptyset$.
Prove or disprove that $\mathbf{J}$ is an equivalence relation.

5. (15 points) For each pair of sets, determine whether they are **disjoint, equal, proper subset** or **none of the above.** Give a justification for each answer.

(a) $[3]_{12}$ and $[2]_8$

(b) $[5]_6$ and $[5]_8$

(c) $[0]_3$ and $[0]_8$

(d) $[3]_5$ and $[13]_{15}$

(e) $[1]_{10}$ and $[1]_{11}$

6. (15 points) Prove that if $a \textbf{ mod } m = c \textbf{ mod } m$ then $a^2 \textbf{ mod } m = c^2 \textbf{ mod } m$.

(Recall that $a \textbf{ mod } m = c \textbf{ mod } m$ means that there exists an integer k such that $a - c = km$.)