CSE 20 Summer 2021
Homework 3
Due date: Sunday, August 22, 2021 at 11:59pm

Topics Induction, Functions, Cardinality

Reading

- Sections 5.1, 5.2, 2.3, 2.5

Key Concepts regular induction, strong induction, structural induction, one-to-one functions, onto functions, bijections, cardinalities.

In this class, unless the instructions explicitly say otherwise, you are required to justify all your answers.

1. (30 points) (6 points each) Prove the following statements using regular or strong induction:

   (a) For all $n \geq 0$,
   $$3|(n^3 - n).$$
   (optional, not for credit. Can you show that $6|(n^3 - n)$ for all $n \geq 0$?)

   (b) For all $n \geq 3$,
   $$\sum_{i=3}^{n} 4^i = \frac{4(4^n - 16)}{3}.$$
   .

   (c) Suppose you only have 4 cent coins and 5 cent coins, Fill in the blanks of the following strong induction proof that you can make change for any integer amount $n \geq 12$.

   Solution: By strong induction on n.
   Basis step: Show that it is true for:
   - $n = 12$ ____(1)____
   - $n = 13$ ____(2)____
   - $n = 14$ ____(3)____
   - $n = 15$ ____(4)____

   Inductive step: Fix $k \geq$ ____(5)____ and assume that,
   for all m with ____(6)____ $\leq m < k$, we can make change for m cents.
   WTS: we can make change for k cents.
   Since ____(7)____ $\leq k-4 < k$, by the Inductive Hypothesis ____(8)____
   Therefore we can make change for k cents using 4 cent coins and 5 cent coins by ____(9)____.

   (d) Define the following sequence of integers recursively as $t_1 = 1, t_2 = 1, \quad t_n = 2t_{n-1} + 2t_{n-2}$ for all $n \geq 3$.
   $$t_n \geq 2^n \text{ for all } n \geq 6.$$

   (e) (problem is optional, not for credit. But still a good exercise for studying.)
   For any $n \geq 2$, let $A_1, \ldots, A_n$ be arbitrary sets. Then
   $$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$
   (you can use the De Morgan's laws for two sets without proof, i.e., that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for any two sets.)

2. (15 points) Define the following sets using a recursive definition:

   (a) The set of all integer multiples of 10. (Don't forget negative integers.)

   (b) The set of all binary strings of the form $\underbrace{101010\ldots10}_{n \text{ copies of } (10)}$ for any positive integer n.

   (c) The set of all full binary trees such that every left child is a leaf.

3. (15 points) (recall the definition of full binary trees from Wednesday's (7/15).)

   For any full binary tree T, let $L(T)$ be the number of leaves of T and let $V(T)$ be the number of vertices of T. Prove using structural induction that $V(T) = 2L(T) - 1$. (Note that the single root vertex is a leaf.)

4. (18 points) (Use the following definitions of $\leq, \geq$, and $=$ that are based on one-to-one and onto functions for the following problems.)

   - $|X| \leq |Y|$ means there exists a function $f : X \to Y$ which is one-to-one.
   - $|X| \geq |Y|$ means there exists a function $f : X \to Y$ which is onto.
   - $|X| = |Y|$ means there exists a function $f : X \to Y$ which is one-to-one and onto (bijection).

   Let A, B and C be arbitrary non-empty sets.

   (a) Suppose that $A \subseteq B$. Show that $|B| \geq |A|$. (Show that there exists an onto function from B to A.)

   (b) Show that $|A| \leq |A \times A|$. (Show that there exists a one-to-one function from A to $A \times A$.)

   (c) (problem is optional, not for credit. But still a good exercise for studying. Really, this is just a corollary to part (a).)

   Show that $|A| \geq |A \cap B|$. (Show that there exists an onto function from A to $A \cap B$.)

5. (18 points)

   (Recall that $f \circ g$ is the *composition of functions f and g* from Thursday's lecture (7/16))

   (a) Prove that for any sets A_1, A_2, A_3, if functions $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$ are both one-to-one then $f_2 \circ f_1 : A_1 \to A_3$ is one-to-one.

   (b) Use induction and the result from part (a) to prove the following claim:

   Claim: For all $n \geq 2$, for any sequence of n functions and any $n + 1$ sets $A_1, \ldots, A_{n+1}$. If

$$
\begin{aligned}
f_1 &: A_1 \to A_2 \\
f_2 &: A_2 \to A_3 \\
\cdots & \\
f_n &: A_n \to A_{n+1}
\end{aligned}
$$

   and that $f_1, f_2, \ldots, f_n$ are all one-to-one, then the function

$$f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1 : A_1 \to A_{n+1}$$

   is one-to-one.